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ADJOINT PROBLEM FOR THE LAPLACE EQUATION UNDER A NONLOCAL BOUNDARY CONDITION

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Abstract. The paper is devoted to a new method of construction of the adjoint problem to a boundary value problem for an elliptic partial differential equation in a bounded domain with nonlocal boundary conditions. In this method, we do not use conditions except the given boundary ones. The method is universal in the sense that it can be applied both to linear ordinary differential equations and to linear partial differential equations.

Keywords: boundary value problem, adjoint problem, second Green's formula, sufficient boundary conditions.

AMS Subject Classification: 35J25.

1. Introduction

It is known that the Naimark method to construct the adjoint problem to a boundary value problem for a linear ordinary differential equation is based on the Lagrange formula related to the initial equation [8]. If the initial equation is of order n, one adds, to the given boundary conditions, some expressions with arbitrary coefficients so that one gets $2n$ conditions. If these $2n$ relations are linearly independent, they make it possible to determine the values of the unknown function as well as the values of its derivatives up to the order $(n-1)$at the edges of the segment. The obtained values are substituted into the Lagrange formula and the result is assumed to be equal to zero. The boundary conditions of the adjoint problem include arbitrary coefficients which appeared in the added relations. Therefore, the boundary conditions of the adjoint problem are not unique. For linear ordinary differential equation of second order this difficulty was eliminated in [1] by a modification of the Naimark method. Later, this new method was applied to a linear ordinary differential equation of the third order [4].The adjoint problem to a boundary value problem for an elliptic linear partial differential equation of the first order with nonlocal boundary condition has been constructed in a unique way (i.e. without ambiguous coefficients) [2]. The solution of the boundary value problem for an ordinary integro-differential equation with local boundary conditions was investigated in [3]. Note that the adjoint problem for elliptic partial differential equations of high order with local boundary conditions were considered in the works of Lions [5], [6], [7].

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2. Formulation of the problem

Let a bounded domain D in the $(x_1,\ x_2)$-plane be convex in the x_2 direction, $D=\{(x_1,x_2):\ :\gamma_1(x_1)\le x_2\le \gamma_2(x_1)\}$, $\Gamma=\partial D$. This means, in particular, that it has a well-defined normal direction at each point. γ_1 and γ_2 are differentiable. Let us consider the following problem:

$$l\,u\equiv \Delta\,u\,(x)=\sum_{k=1}^{2}\frac{\partial^2 u\,(x)}{\partial\,x_k^2}=0,\qquad x=(x_1,x_2)\in D\subset R^2, \tag{1}$$

$$\sum_{s=1}^{2}\left[\sum_{k=1}^{2}\alpha_{m\,k}^{(s)}(x_1)\,\frac{\partial\,u\,(x)}{\partial\,x_k}+\alpha_m^{(s)}(x_1)\,u\,(x)\right]\Bigg|_{x_2=\gamma_s(x_1)}=0\,,\quad m=1,2\,,\quad x_1\in[a_1,\ b_1]\,. \tag{2}$$

Here, $\alpha_{m\,k}^{(s)}(x_1)$ and $\alpha_m^{(s)}(x_1)$, $m,\ k,\ s=1,\ 2$,$x_1\in[a_1,\ b_1]$, are continuous functions, conditions (2) are linearly independent. Let us construct the second Green formula for the equation (1)

$$(l u,\ v)=\sum_{k=1}^{2}\int\limits_D\frac{\partial^2 u\,(x)}{\partial\,x_k^2}\,\bar v\,(x)\,dx=\sum_{k=1}^{2}\int\limits_\Gamma\left[\frac{\partial\,u\,(x)}{\partial\,x_k}\ \ \bar v\,(x)-u\,(x)\ \ \frac{\partial\,\bar v\,(x)}{\partial\,x_k}\right]\ \times$$

$$\times\ \cos\,(n,\ x_k)\ dx+(u,\ l^*v),$$

where n- is the external normal to Γ,

$$l^*v=\Delta\,v(x)=\sum_{k=1}^{2}\frac{\partial^2 v\,(x)}{\partial\,x_k^2}\ ,\qquad x\in D. \tag{3}$$

Let us transform the bilinear expression contained in Green's formula

$$B\,(u,\ v)=\sum_{k=1}^{2}\int\limits_\Gamma\left[\frac{\partial\,u\,(x)}{\partial\,x_k}\ \ \bar v\,(x)-u\,(x)\ \ \frac{\partial\,\bar v\,(x)}{\partial\,x_k}\right]\cdot\cos\,(n,\ x_k)\ dx=$$

$$=\int\limits_{a_1}^{b_1}\left\{\left[\frac{\partial\,u\,(x)}{\partial\,x_1}\bar v\,(x)-u\,(x)\frac{\partial\,\bar v\,(x)}{\partial\,x_1}\right]\Bigg|_{x_2=\gamma_1(x_1)}\cdot\cos\,(n,\ x_1)\,+\right.$$

$$+\left[\frac{\partial\,u\,(x)}{\partial\,x_2}\bar v\,(x)-u\,(x)\frac{\partial\,\bar v\,(x)}{\partial\,x_2}\right]\Bigg|_{x_2=\gamma_1(x_1)}\cdot\cos\,(n,\ x_2)\,+$$

$$+\left[\frac{\partial\,u\,(x)}{\partial\,x_1}\bar v\,(x)-u\,(x)\frac{\partial\,\bar v\,(x)}{\partial\,x_1}\right]\Bigg|_{x_2=\gamma_2(x_1)}\cdot\cos\,(n,\ x_1)\,+$$

$$\left.+\left[\frac{\partial\,u\,(x)}{\partial\,x_2}\bar v\,(x)-u\,(x)\frac{\partial\,\bar v\,(x)}{\partial\,x_2}\right]\Bigg|_{x_2=\gamma_2(x_1)}\cdot\cos\,(n,\ x_2)\right\}\ \frac{dx_1}{\cos\,(x_1,\ \tau)}=0. \tag{4}$$

Here τ is the tangential direction to the boundary Γ.

One has:

$$u'(x_1,\ \gamma_k(x_1))=\frac{\partial\,u\,(x)}{\partial\,x_1}\Bigg|_{x_2=\gamma_k(x_1)}+\frac{\partial\,u\,(x)}{\partial\,x_2}\Bigg|_{x_2=\gamma_k(x_1)}\gamma_k'(x_1),k=1,\ 2. \tag{5}$$

Thus, the expression (4) can be written as follows:

$$B\,(u,\,v) \equiv \int\limits_{a_1}^{b_1} \left\{ \bar{v}\,(x_1,\,\gamma_1(x_1)) \,\,\left[u'(x_1,\,\gamma_1(x_1)) - \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} \gamma\prime_1(x_1) \right] - \right.$$

$$\left. -u\,(x_1,\,\gamma_1(x_1))\,\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_1}\right|_{x_2=\gamma_1(x_1)} \right\} \gamma\prime_1(x_1) dx_1 -$$

$$-\int\limits_{a_1}^{b_1} \left\{ \bar{v}\,(x_1,\,\gamma_1(x_1))\,\left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} - u\,(x_1,\,\gamma_1(x_1))\,\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} \right\} dx_1 -$$

$$-\int\limits_{a_1}^{b_1} \left\{ \bar{v}\,(x_1,\,\gamma_2(x_1)) \,\,\left[u\prime(x_1,\,\gamma_2(x_1)) - \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} \gamma\prime_2(x_1) \right] - \right.$$

$$\left. -u\,(x_1,\,\gamma_2(x_1))\,\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_1}\right|_{x_2=\gamma_2(x_1)} \right\} \gamma\prime_2(x_1)\; dx_1 +$$

$$+\int\limits_{a_1}^{b_1} \left\{ \bar{v}\,(x_1,\,\gamma_2(x_1))\,\left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} - u\,(x_1,\,\gamma_2(x_1))\,\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} \right\}\; dx_1 =$$

$$= \bar{v}\,(b_1,\,\gamma_1(b_1))\;u\,(b_1,\,\gamma_1(b_1)) \cdot \gamma\prime_1(b_1) - \bar{v}\,(a_1,\,\gamma_1(a_1))\;u\,(a_1,\,\gamma_1(a_1)) \cdot \gamma\prime_1(a_1) -$$

$$-\int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_1(x_1))\;\left[\gamma\prime_1(x_1)\,\bar{v}\,(x_1,\,\gamma_1(x_1))\right]'\; dx_1 -$$

$$-\int\limits_{a_1}^{b_1} \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} \bar{v}\,(x_1,\,\gamma_1(x_1))\,\gamma\prime_1^2(x_1)\, dx_1 - \int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_1(x_1))\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_1}\right|_{x_2=\gamma_1(x_1)} \gamma\prime_1(x_1) dx_1 -$$

$$-\int\limits_{a_1}^{b_1} \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} \bar{v}\,(x_1,\,\gamma_1(x_1))\; dx_1 + \int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_1(x_1))\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_1(x_1)} dx_1 -$$

$$-\bar{v}\,(b_1,\,\gamma_2(b_1))\;u\,(b_1,\,\gamma_2(b_1)) \cdot \gamma\prime_2(b_1) + \bar{v}\,(a_1,\,\gamma_2(a_1))\;u\,(a_1,\,\gamma_2(a_1)) \cdot \gamma\prime_2(a_1) +$$

$$+\int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_2(x_1))\;\left[\gamma\prime_2(x_1)\,\bar{v}\,(x_1,\,\gamma_2(x_1))\right]'\; dx_1 +$$

$$+\int\limits_{a_1}^{b_1} \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} \bar{v}\,(x_1,\,\gamma_2(x_1))\,\gamma\prime_2^2(x_1)\, dx_1 + \int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_2(x_1))\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_1}\right|_{x_2=\gamma_2(x_1)} \gamma\prime_2(x_1) dx_1 +$$

$$+\int\limits_{a_1}^{b_1} \left.\frac{\partial\, u\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} \bar{v}\,(x_1,\,\gamma_2(x_1))\; dx_1 - \int\limits_{a_1}^{b_1} u\,(x_1,\,\gamma_2(x_1))\left.\frac{\partial\,\bar{v}\,(x)}{\partial\, x_2}\right|_{x_2=\gamma_2(x_1)} dx_1 =$$

$$
=\int\limits_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} \left[-\bar{v}(x_1,\gamma_1(x_1))\gamma\prime_1^2(x_1)-\bar{v}(x_1,\gamma_1(x_1))\right] dx_1+
$$

$$
+\int\limits_{a_1}^{b_1} \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} \left[\bar{v}(x_1,\gamma_2(x_1))\gamma\prime_2^2(x_1)+\bar{v}(x_1,\gamma_2(x_1))\right] dx_1+
$$

$$
+\int\limits_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\left[-\bar{v}\prime(x_1,\gamma_1(x_1))\gamma\prime_1(x_1)-\gamma\prime\prime_1(x_1)\bar{v}(x_1,\gamma_1(x_1))-\left.\frac{\partial \bar{v}(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)}\gamma\prime_1(x_1)+\right.
$$

$$
\left.+\left.\frac{\partial \bar{v}(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\right] dx_1+\int\limits_{a_1}^{b_1} u(x_1,\gamma_2(x_1))\left[-\bar{v}\prime(x_1,\gamma_2(x_1))\gamma\prime_2(x_1)+\gamma\prime\prime_2(x_1)\bar{v}(x_1,\gamma_2(x_1))+\right.
$$

$$
\left.+\left.\frac{\partial \bar{v}(x)}{\partial x_1}\right|_{x_2=\gamma_2(x_1)}\gamma\prime_2(x_1)-\left.\frac{\partial \bar{v}(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\right] dx_1=0, \tag{6}
$$

here

$$
u(b_1,\gamma_1(b_1))= u(a_1,\gamma_1(a_1))=u(a_1,\gamma_2(a_1))=u(b_1,\gamma_2(b_1))=0. \tag{7}
$$

Coming back to the boundary conditions (2), let us multiply them by the $v_m(x_1)$. We get

$$
\sum_{m=1}^{2}\sum_{s=1}^{2}\left[\sum_{k=1}^{2}\int\limits_{a_1}^{b_1}\alpha_{mk}^{(s)}(x_1)\left.\frac{\partial u(x)}{\partial x_k}\right|_{x_2=\gamma_s(x_1)}\bar{v}_m(x_1)\,dx_1+\right.
$$

$$
\left.+\int\limits_{a_1}^{b_1}\alpha_m^{(s)}(x_1)\,u(x_1,\gamma_s(x_1))\,\bar{v}_m(x_1)\,dx_1\right]=0.
$$

Taking (5) into account, we get:

$$
\sum_{m=1}^{2}\sum_{s=1}^{2}\left\{\int\limits_{a_1}^{b_1}\alpha_{m1}^{(s)}(x_1)\,\bar{v}_m(x_1)\left[u'(x_1,\gamma_s(x_1))-\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_s(x_1)}\gamma\prime_s(x_1)\right] dx_1+\right.
$$

$$
\left.+\int\limits_{a_1}^{b_1}\alpha_{m2}^{(s)}(x_1)\,\bar{v}_m(x_1)\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_s(x_1)}dx_1+\int\limits_{a_1}^{b_1}\alpha_m^{(s)}(x_1)\,u(x_1,\gamma_s(x_1))\,\bar{v}_m(x_1)dx_1\right\}=0,
$$

or

$$
\sum_{s=1}^{2}\sum_{m=1}^{2}\int\limits_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_s(x_1)}\left[\alpha_{m2}^{(s)}(x_1)\,\bar{v}_m(x_1)-\alpha_{m1}^{(s)}(x_1)\,\bar{v}_m(x_1)\,\gamma\prime_s(x_1)\right] dx_1+
$$

$$
+\sum_{s=1}^{2}\sum_{m=1}^{2}\int\limits_{a_1}^{b_1}u(x_1,\gamma_s(x_1))\left[\alpha_m^{(s)}(x_1)\,\bar{v}_m(x_1)-\left(\alpha_{m1}^{(s)}(x_1)\,\bar{v}_m(x_1)\right)'\right] dx_1=0. \tag{8}
$$

Here the conditions (7) are applied.

Remark 2.1. The functions $v_1(x_1)$ and $v_2(x_1)$ can be choosen so that the coefficients in the

square brackets at two fixed functions from $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}$, $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_s(x_1)}$, $u(x_1, \gamma_1(x_1))$and $u(x_1, \gamma_2(x_1))$ conincide with the corresponding coefficients in (6). The integrals corresponding to these two summands are determined from (8) and substituted in (6). We obtain one expression depending on the remaining two functions from $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}$, $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_s(x_1)}$, $u(x_1, \gamma_1(x_1))$and $u(x_1, \gamma_2(x_1))$. This expression is equal to zero independently of these two functions. Here we shall compare the coefficients at $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}$ and $u(x_1, \gamma_2(x_1))$.
Thus, the functions $v_m(x_1)$ are determined from the equations:

$$\sum_{m=1}^{2}\left[\alpha_{m2}^{(1)}(x_1)\ -\alpha_{m1}^{(1)}(x_1)\ \gamma\prime_1(x_1)\right]\ \bar{v}_m(x_1)=-\bar{v}(x_1,\,\gamma_1(x_1))\gamma\prime_1^2(x_1)-\bar{v}(x_1,\,\gamma_1(x_1)),$$

$$\sum_{m=1}^{2}\left[\alpha_{m}^{(2)}(x_1)\,\bar{v}_m(x_1)\ -\left(\alpha_{m1}^{(2)}(x_1)\,\bar{v}_m(x_1)\right)'\right]\ =\bar{v}\prime(x_1,\,\gamma_2(x_1))\gamma\prime_2(x_1)+\bar{v}(x_1,\,\gamma_2(x_1))\gamma\prime\prime_2(x_1)+$$

$$+\left.\frac{\partial\,\bar{v}\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_2(x_1)}\gamma\prime_2(x_1)-\left.\frac{\partial\,\bar{v}\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_2(x_1)}. \tag{9}$$

From the first equation under the condition

$$\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)\neq 0, \tag{10}$$

we obtain $\bar{v}_1(x_1)$and $\bar{v}_2(x_1)$:

$$\bar{v}_2(x_1)=-\frac{\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)}{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\,\bar{v}_1(x_1)-\frac{\left[1+\gamma\prime_1^2(x_1)\right]\ \bar{v}(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\ \gamma_1'(x_1)}. \tag{11}$$

Substituting (11) in the second equation (9), we get:

$$\alpha_1^{(2)}(x_1)\ \bar{v}_1(x_1)-\alpha_2^{(2)}(x_1)\frac{\left[\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)\right]\ \bar{v}_1(x_1)+\left[1+\gamma\prime_1^2(x_1)\right]\ \bar{v}(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}-$$

$$-\left(\alpha_{11}^{(2)}(x_1)\ \bar{v}_1(x_1)\right)'+$$

$$+\left\{\alpha_{21}^{(2)}(x_1)\ \frac{\left[\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)\right]\bar{v}_1(x_1)+\left[1+\gamma\prime_1^2(x_1)\right]\ \bar{v}(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\right\}'=$$

$$=\bar{v}\prime(x_1,\,\gamma_2(x_1))\ \gamma\prime_2(x_1)+$$

$$+\bar{v}(x_1,\,\gamma_2(x_1))\ \gamma\prime\prime_2(x_1)+\left.\frac{\partial\,\bar{v}\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_2(x_1)}\gamma\prime_2(x_1)-\left.\frac{\partial\,\bar{v}\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_2(x_1)}. \tag{12}$$

Thus, we obtain the linear equation (12) containing the first order derivative of the function $\bar{v}_1(x_1)$. The solution of this equation may be constructed in an analytic form. For this we need to know one solution of the non-homogeneous equation. The equation (12) may be written in the following way:

$$A(x_1)\ \bar{v}_1'(x_1)+B(x_1)\ \bar{v}_1(x_1)=F(x_1)\ , \tag{13}$$

where

$$A(x_1) = -\alpha_{11}^{(2)}(x_1) + \alpha_{21}^{(2)}(x_1)\frac{\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)},$$

$$B(x_1) = \alpha_1^{(2)}(x_1) \ - \alpha_2^{(2)}(x_1)\frac{\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)} - \alpha_{11}^{(2)}\prime(x_1) +$$
$$+\left\{\alpha_{21}^{(2)}(x_1)\ \frac{\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma_1'(x_1)}\right\}',$$

$$F(x_1) = \alpha_2^{(2)}(x_1)\frac{\left[1+\gamma\prime_1^2(x_1)\right]\ \bar v(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)} - \left\{\alpha_{21}^{(2)}(x_1)\ \frac{\left[1+\gamma\prime_1^2(x_1)\right]\ \bar v(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)}\right\}' -$$

$$\bar v'(x_1,\,\gamma_2(x_1))\,\gamma\prime_2(x_1) + \bar v(x_1,\,\gamma_2(x_1))\,\gamma\prime\prime_2(x_1) + \left.\frac{\partial\,\bar v\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_2(x_1)}\gamma\prime_2(x_1) - \left.\frac{\partial\,\bar v\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_2(x_1)}.$$

If we have

$$A(x_1) \neq 0, \tag{14}$$

then we get (13):

$$\bar v_1'(x_1) = -\frac{B(x_1)}{A(x_1)}\ \bar v_1(x_1) + \frac{F(x_1)}{A(x_1)},$$

where

$$\bar v_1(x_1) = c\ e^{-\int\limits_{a_1}^{x_1}\frac{B(t)}{A(t)}\,dt} + \int\limits_{a_1}^{x_1}\frac{F(\tau)}{A(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{B(t)}{A(t)}\,dt}d\tau,$$

is the general solution of (13). Assuming $c = 0$ one gets the solution

$$\bar v_1(x_1) = \int\limits_{a_1}^{x_1}\frac{F(\tau)}{A(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{B(t)}{A(t)}\,dt}d\tau. \tag{15}$$

If we substitute this expression in (11), we get:

$$\bar v_2(x_1) = -\frac{\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\,\gamma_1'(x_1)}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma_1'(x_1)}\int\limits_{a_1}^{x_1}\frac{F(\tau)}{A(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{B(t)}{A(t)}\,dt}d\tau -$$
$$-\frac{\left[1+\gamma\prime_1^2(x_1)\right]\ \bar v(x_1,\,\gamma_1(x_1))}{\alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)}. \tag{16}$$

Taking into account (9) and substituting (15) and (16) in (8), we get :

$$-\int\limits_{a_1}^{b_1}(1+\gamma\prime_1^2(x_1))\ \bar v(x_1,\,\gamma_1(x_1))\ \left.\frac{\partial\,u\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_1(x_1)}dx_1 +$$
$$+\int\limits_{a_1}^{b_1}\left[\bar v\prime(x_1,\,\gamma_2(x_1))\ \gamma\prime_2(x_1) +\ \bar v(x_1,\,\gamma_2(x_1))\,\gamma\prime\prime_2(x_1) +\right.$$

$$+\left.\frac{\partial \bar{v}(x)}{\partial x_1}\right|_{x_2=\gamma_2(x_1)} \gamma\prime_2(x_1) - \left.\left.\frac{\partial \bar{v}(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\right] u(x_1,\, \gamma_2(x_1))\, dx_1 =$$

$$= -\sum_{m=1}^{2}\int_{a_1}^{b_1}\left[\alpha_{m2}^{(2)}(x_1) - \alpha_{m1}^{(2)}(x_1)\,\gamma\prime_2(x_1)\right]\,\bar{v}_m(x_1)\,\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} dx_1 -$$

$$-\sum_{m=1}^{2}\int_{a_1}^{b_1}\left[\alpha_m^{(1)}(x_1)\,\bar{v}_m(x_1) - \left(\alpha_{m1}^{(1)}(x_1)\,\bar{v}_m(x_1)\right)'\right]\, u(x_1,\, \gamma_1(x_1))\, dx_1.$$

Finally, substituting this expression in (6), we get:

$$B(u,\, v) = -\sum_{m=1}^{2}\int_{a_1}^{b_1}\left[\alpha_{m2}^{(2)}(x_1) - \alpha_{m1}^{(2)}(x_1)\,\gamma\prime_2(x_1)\right]\,\bar{v}_m(x_1)\,\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} dx_1 -$$

$$-\sum_{m=1}^{2}\int_{a_1}^{b_1}\left[\alpha_m^{(1)}(x_1)\,\bar{v}_m(x_1) - \left(\alpha_{m1}^{(1)}(x_1)\,\bar{v}_m(x_1)\right)'\right]\, u(x_1,\, \gamma_1(x_1))\, dx_1 +$$

$$+\int_{a_1}^{b_1}(1+\gamma\prime_2^2(x_1))\,\bar{v}(x_1,\,\gamma_2(x_1))\,\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} dx_1 +$$

$$+\int_{a_1}^{b_1}\left[-\bar{v}'(x_1,\,\gamma_1(x_1))\,\gamma\prime_1(x_1) - \bar{v}(x_1,\,\gamma_1(x_1))\,\gamma\prime\prime_1(x_1) -\right.$$

$$\left. -\left.\frac{\partial \bar{v}(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)}\gamma\prime_1(x_1) + \left.\frac{\partial \bar{v}(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\right] u(x_1,\,\gamma_1(x_1))\, dx_1 = 0.$$

Since the equation holds for arbitrary $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}$ and $u(x_1,\,\gamma_1(x_1))$, we get the following boundary condition for the adjoint problem:

$$-\sum_{m=1}^{2}\left[\bar{\alpha}_{m2}^{(2)}(x_1) - \bar{\alpha}_{m1}^{(2)}(x_1)\,\gamma\prime_2(x_1)\right]\, v_m(x_1) + (1+\gamma\prime_2{}^2(x_1))v(x_1,\,\gamma_2(x_1)) = 0\,,$$

$$-\sum_{m=1}^{2}\left[\bar{\alpha}_m^{(1)}(x_1)\, v_m(x_1) - \left(\bar{\alpha}_{m1}^{(1)}(x_1)\, v_m(x_1)\right)'\right] - v'(x_1,\,\gamma_1(x_1))\,\gamma\prime_1(x_1) -$$

$$-v(x_1,\,\gamma_1(x_1))\,\gamma\prime\prime_1(x_1) - \left.\frac{\partial v(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)}\gamma\prime_1(x_1) + \left.\frac{\partial v(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} = 0. \qquad (17)$$

Substituting $v_1(x_1)$ and $v_2(x_1)$ from (15) and (16) in (17), we get:

$$-\left[\bar{\alpha}_{12}^{(2)}(x_1) - \bar{\alpha}_{11}^{(2)}(x_1)\,\gamma\prime_2(x_1)\right]\int_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\, e^{-\int_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}\, d\tau +$$

$$+\left[\bar{\alpha}_{22}^{(2)}(x_1) - \bar{\alpha}_{21}^{(2)}(x_1)\,\gamma\prime_2(x_1)\right]\left\{\frac{\left[1+\gamma\prime_1^2(x_1)\right]\, v(x_1,\,\gamma_1(x_1))}{\bar{\alpha}_{22}^{(1)}(x_1) - \bar{\alpha}_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)} + \right.$$

$$\left. + \frac{\bar{\alpha}_{12}^{(1)}(x_1) - \bar{\alpha}_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1) - \bar{\alpha}_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)}\int_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\, e^{-\int_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}\, d\tau\right\} +$$

$$+\left[1+\gamma\prime_2^2(x_1)\right]\ v(x_1,\,\gamma_2(x_1))=0, \tag{18}$$

$$-\bar{\alpha}_1^{(1)}(x_1)\int\limits_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}d\tau+\bar{\alpha}_2^{(1)}(x_1)\ \left\{\frac{\left[1+\gamma\prime_1^2(x_1)\right]\ v(x_1,\,\gamma_1(x_1))}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma_1'(x_1)}+\right.$$

$$\left.+\frac{\bar{\alpha}_{12}^{(1)}(x_1)-\bar{\alpha}_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\ \int\limits_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}d\tau\right\}+$$

$$+\left\{\bar{\alpha}_{11}^{(1)}(x_1)\int\limits_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}d\tau-\ \bar{\alpha}_{21}^{(1)}(x_1)\left[\frac{\left[1+\gamma\prime_1^2(x_1)\right]\ v(x_1,\,\gamma_1(x_1))}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma_1'(x_1)}+\right.\right.$$

$$\left.\left.+\frac{\bar{\alpha}_{12}^{(1)}(x_1)-\bar{\alpha}_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\ \int\limits_{a_1}^{x_1}\frac{\bar{F}(\tau)}{\bar{A}(\tau)}\ e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}d\tau\right]\right\}'-\ v'(x_1,\,\gamma_1(x_1))\ \gamma_1'(x_1)-$$

$$-\,v(x_1,\,\gamma_1(x_1))\ \gamma\prime\prime_1(x_1)-\ \frac{\partial\,v\,(x)}{\partial\,x_1}\bigg|_{x_2=\gamma_1(x_1)}\gamma\prime_1(x_1)+\ \frac{\partial\,v\,(x)}{\partial\,x_2}\bigg|_{x_2=\gamma_1(x_1)}=0. \tag{19}$$

The conditions (18) and (19) can be rewritten in the following way:

$$\frac{\bar{\alpha}_{22}^{(2)}(x_1)-\bar{\alpha}_{21}^{(2)}(x_1)\ \gamma\prime_2(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\left[1+\gamma\prime_1^2(x_1)\right]\ v\,(x_1,\,\gamma_1(x_1))\ +\left[1+\gamma\prime_2^2(x_1)\right]\ v\,(x_1,\,\gamma_2(x_1))\,+$$

$$+\left\{\frac{\bar{\alpha}_{22}^{(2)}(x_1)-\bar{\alpha}_{21}^{(2)}(x_1)\ \gamma\prime_2(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\left[\bar{\alpha}_{12}^{(1)}(x_1)-\bar{\alpha}_{11}^{(1)}(x_1)\ \gamma\prime_1(x_1)\right]-\right.$$

$$\left.-\left[\bar{\alpha}_{12}^{(2)}(x_1)-\bar{\alpha}_{11}^{(2)}(x_1)\ \gamma\prime_2(x_1)\right]\right\}\cdot\int\limits_{a_1}^{x_1}\ e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}\ \frac{d\tau}{\bar{A}(\tau)}\times$$

$$\times\left\{\left[\frac{\bar{\alpha}_{2}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau)-\bar{\alpha}_{21}^{(1)}(\tau)\ \gamma_1(\tau)}\ \left[1+\gamma\prime_1^2(\tau)\right]-\left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau)-\bar{\alpha}_{21}^{(1)}(\tau)\ \gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)'\right]\times\right.$$

$$\times v\,(\tau,\,\gamma_1(\tau))\ +\gamma\prime_2(\tau)\,v\prime\,(\tau_1,\,\gamma_2(\tau))\ -\left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau)-\bar{\alpha}_{21}^{(1)}(\tau)\ \gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)\times$$

$$\times v\prime\,(\tau,\,\gamma_1(\tau))\ +\gamma\prime\prime_2(\tau)\,v\,(\tau,\,\gamma_2(\tau))+\ \frac{\partial\,v\,(\tau,x_2)}{\partial\,\tau}\bigg|_{x_2=\gamma_2(\tau)}\gamma\prime_2(\tau)-$$

$$\left.-\ \frac{\partial\,v\,(\tau,x_2)}{\partial\,x_2}\bigg|_{x_2=\gamma_2(\tau)}\right\}=0, \tag{20}$$

$$\left\{\ \frac{\bar{\alpha}_{2}^{(1)}(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\ \left[1+\gamma\prime_1^2(x_1)\right]-\left(\frac{\bar{\alpha}_{21}^{(1)}(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\left[1+\gamma\prime_1^2(x_1)\right]\right)'-\right.$$

$$\left.-\ \gamma\prime\prime_1(x_1)\right\}\ v\,(x_1,\,\gamma_1(x_1))\ -\left\{\gamma\prime_1(x_1)+\frac{\bar{\alpha}_{21}^{(1)}(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1)-\bar{\alpha}_{21}^{(1)}(x_1)\ \gamma\prime_1(x_1)}\left[1+\gamma\prime_1^2(x_1)\right]\right\}\times$$

$$\times v\prime(x_1,\,\gamma_1(x_1)) + \left[\bar{\alpha}_2^{(1)}(x_1)\frac{\bar{\alpha}_{12}^{(1)}(x_1) - \bar{\alpha}_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1) - \bar{\alpha}_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)} - \bar{\alpha}_1^{(1)}(x_1)\right]\int\limits_{a_1}^{x_1} e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}\,\frac{d\tau}{\bar{A}(\tau)}\times$$

$$\times\left\{\left[\frac{\bar{\alpha}_2^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma\prime_1(\tau)}\,\left[1+\gamma\prime_1^2(\tau)\right] - \left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)'\times\right.\right.$$

$$\times v\,(\tau,\,\gamma_1(\tau))\;+\gamma\prime_2(\tau)v'\,(\tau,\,\gamma_2(\tau))\;-\left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)\times$$

$$\times v\prime\,(\tau,\,\gamma_1(\tau))\;+\gamma\prime\prime_2(\tau)\,v(\tau,\,\gamma_2(\tau)) + \left.\frac{\partial\,v\,(\tau,x_2)}{\partial\,\tau}\right|_{x_2=\gamma_2(\tau)}\gamma\prime_2(\tau)-$$

$$\left.\left.-\left.\frac{\partial\,v\,(\tau,x_2)}{\partial\,x_2}\right|_{x_2=\gamma_2(\tau)}\right\} + \left\{\left[\bar{\alpha}_{11}^{(1)}(x_1) - \bar{\alpha}_{21}^{(1)}(x_1)\frac{\bar{\alpha}_{12}^{(1)}(x_1) - \bar{\alpha}_{11}^{(1)}(x_1)\,\gamma\prime_1(x_1)}{\bar{\alpha}_{22}^{(1)}(x_1) - \bar{\alpha}_{21}^{(1)}(x_1)\,\gamma\prime_1(x_1)}\right]\times\right.$$

$$\times\int\limits_{a_1}^{x_1} e^{-\int\limits_{\tau}^{x_1}\frac{\bar{B}(t)}{\bar{A}(t)}\,dt}\,\frac{d\tau}{\bar{A}(\tau)}\,\left\{\left[\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma_1(\tau)}\,\left[1+\gamma\prime_1^2(\tau)\right] -\right.\right.$$

$$-\left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)'\;v\,(\tau,\,\gamma_1(\tau))\;+\gamma\prime_2(\tau)\,v'\,(x_1,\,\gamma_2(\tau))-$$

$$-\left(\frac{\bar{\alpha}_{21}^{(2)}(\tau)}{\bar{\alpha}_{22}^{(1)}(\tau) - \bar{\alpha}_{21}^{(1)}(\tau)\,\gamma\prime_1(\tau)}\left[1+\gamma\prime_1^2(\tau)\right]\right)\times v\prime(\tau,\,\gamma_1(\tau))\;+\gamma\prime\prime_2(\tau)\,v\,(\tau,\,\gamma_2(\tau))+$$

$$\left.\left.\left.+\left.\frac{\partial\,v\,(\tau,x_2)}{\partial\,\tau}\right|_{x_2=\gamma_2(\tau)}\gamma\prime_2(\tau) - \left.\frac{\partial\,v\,(\tau,x_2)}{\partial\,x_2}\right|_{x_2=\gamma_2(\tau)}\right\}\right\}' - \left.\frac{\partial\,v\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_1(x_1)}\gamma\prime_1(x_1)+$$

$$+\left.\frac{\partial\,v\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_1(x_1)} = 0. \tag{21}$$

Thus, we obtain the following statement:

Theorem 2.1. *If the domain $D = \{(x_1, x_2) : \gamma_1(x_1) \le x_2 \le \gamma_2(x_1)\}$ is bounded, the functions $\gamma_1(x_1)$ and $\gamma_2(x_1)$ are two times differentiable, the boundary conditions* (2) *are linear independent, the coefficients in these conditions satisfy*

$$\alpha_{mk}^{(1)}(x_1) \in C^{(1)}(a_1,\;b_1),\;\alpha_{m1}^{(2)}(x_1) \in C^{(1)}(a_1,\;b_1),\;m,\;k=1,\;2;\alpha_{m2}^{(2)}(x_1) \in C\,(a_1,\;b_1),$$
$$\alpha_m^{(s)}(x_1) \in C^{(1)}(a_1,\;b_1),\;m,\;s=1,\;2;$$

and (7), (10) *and* (14) *hold, then the problem adjoint to* (1)-(2) *is given by* (3), (20), (21).

Remark 2.2. From (20) and (21) one can see that the boundary conditions of the adjoint problem contain both non-local and global terms. In other words, these conditions contain $v\,(x_1,\,\gamma_1(x_1)), v\,(x_1,\,\gamma_2(x_1)), \left.\frac{\partial\,v\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_1(x_1)}, \left.\frac{\partial\,v\,(x)}{\partial\,x_1}\right|_{x_2=\gamma_2(x_1)}, \left.\frac{\partial\,v\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_1(x_1)}, \left.\frac{\partial\,v\,(x)}{\partial\,x_2}\right|_{x_2=\gamma_2(x_1)}$ both in the integrand (global terms) and outside of the integrals (non-local terms).

References

[1] Aliyev, N.A, Guliyeva, A.M., (2013), Building a problem adjoint in a new sense to the boundary value problem for a constant coefficient, second order ordinary linear differential equation, Transactions of Azerbaijan National Academy of Sciences, Informatics and control problems, 33(3), pp.44-50.
[2] Aliyev, N.A, Guliyeva, A.M., (2013), Construction of adjoint problem for equation Cauchy-Riemaun, News of Baku University, Physico-mathematical sciences, 4, pp.43-54.
[3] Alkan, S., Secer, A., (2015), Solution of nonlinear fractional boundary value problem with nonhomogeneous boundary conditions, Applied and Computational Mathematics, 14(3), pp.284-295.
[4] Guliyeva, A.M., (2013), Construction of a problem adjoint in a new approach to the boundary value problem for the third order linear differential equation, Proceedings of IAM, 2(2), pp.216-223.
[5] Lions, J.L., Magenes, E., (1972), Non-Homogeneous Boundary Value Problems and Applications, V.I, Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg, 357p.
[6] Lions, J.L., (1969), Quelques Methodes de Resolution des Problemes Aux Limites Non Lineaires, (French) Dunod, Gauthier-Villars, Paris, 554p.
[7] Lions, J.L., (1961), Equations Differentielles Operationnelles et Problemes Aux Limites, Springer-Verlag Berlin Heidelberg GMBH, 292p.
[8] Naimark, M.A., (1967), Linear Differential Operators, Part I: Elementary Theory of Linear Differential Operators, Frederick Ungar Publishing Co., New York, 144p.

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